FRACTURING ROCK WITH ULTRA HIGH PRESSURE WATER

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Abstract

Modelling issues are considered for the process of cracking rock in mines using ultra high pressure water. The elevated pressures are caused by the ignition of a propellant and may be as large as 1000MPa. We first consider time, length and pressure scales and then derive a model for the propagation of a two-dimensional crack. A number of aspects of this model are considered and similarity solutions and behaviour near the crack tip are investigated. Consideration is given to a simplified model where the elastic component of the interaction between the rock and the fluid is handled using an elementary closure law: in this case much progress may be made and closed-form solutions may be determined. Conditions are also identified where a model based on "impulsive" lubrication theory is appropriate. However, this leads to a very challenging problem. Finally, some other ways of extending the model to include (for example) fluid leak-off into the rock are discussed.

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1 Introduction

Potentially fatal respiratory diseases such as mesothelioma and pneumonoultramicroscopicsilicovolcanoconiosis are a constant hazard in mining. One of the principal causes of these and other serious diseases is the presence of small solid particles (radius $\sim 1\mu$m) in the working atmosphere. Particles of this size are particularly harmful to the lungs as they are too small to be ejected by coughing, but too large to pass through the alveoli. The continued presence in the lungs and carcinogenic nature of these particles therefore presents a major health hazard to miners.

One popular method used to create workable material in mines consists of the somewhat unsubtle tactic of simply initiating an explosion (using dynamite, for example) in portions of as yet unmined rock. In this fashion the rock is broken up and the end product can eventually be harvested. Unfortunately, mining carried out in this way suffers from at least three shortcomings, namely (a) the explosion creates solid rock particles of all sizes, including those of the dangerous dimensions described above, (b) the destruction involved is excessive and a great deal of chemical energy is wasted, and (c) large areas of the mine must be cleared of personnel during the explosion process.

This report investigates some aspects of an alternative method for cracking rock in which ultra high pressure water is used to open fissures in the rock. The high pressure is delivered by detonating a propellant charge in a water-filled hole in the rock. Many standard nitrocellulose-based propellants or similar explosives may be used for this purpose. The hope is that, instead of being literally blown to pieces, planar cracks will form to cut the rock into slices. Little chemical energy is wasted in this process, few very small rock particles are produced and the event is controllable enough for the mine not to have to be completely evacuated as it takes place.

Although using propellant and water in this manner to crack rock is an attractive idea, a number of important modelling questions need to be addressed before the process, if seriously contemplated for adoption, can be scientifically analysed. These include:

- At the ultra high pressures envisaged, will the water even remain liquid? If so, can it still be treated as incompressible?
- Is it possible to construct a reasonably tractable mathematical model for the process of rock cracking in this manner?
- Are any model simplifications possible? If so, do these give us any idea

of the order of magnitude of the crack dimensions and speeds that are likely to be produced by the process?

- Is it possible that "impulsive lubrication theory" will be required to model the process because of the particularly high pressures that are involved?

Each of these matters will be addressed in the sections below.

2 General details of the process

We first consider the parameter values and ranges that are envisaged. Though clearly when propellant is ignited a great number of individual energetic processes take place, we have some idea of the orders of magnitude of the physical parameters that are relevant. We anticipate that the maximum pressures produced will lie in the range $250-1000$MPa, the cracking will take place over a (deflagration rather than a detonation) timescale of duration $0.5-2$ms with a crack propagation speed of $20-200$m/s.[1] We assume that the cracks will spread out from a water-filled borehole of typical dimensions 1m long with a diameter of 40mm, the water being driven into the rock by the ignition of a nitrocellulose based propellant or similar. It is not clear what it might be reasonable to assume regarding pre-existing cracks in the rock, and it is also not clear what part vapour might play in opening the crack. In the work below, we shall therefore assume that the crack is filled only with water. Clearly the role of pre-existing cracks could be a key one, and this is a topic that would repay further study.

2.1 Properties of water at working conditions

Since the whole cracking process relies on using water to break up rock, we first briefly consider how water might behave at such high pressures. The phase diagram of water (Figure 1) is exceedingly complicated, but shows that even for fairly modest temperatures (400-600K) water will remain liquid even at pressures as large as 10^{10}Pa. Though we have no prior information concerning the temperatures at which the whole process takes place, it seems fairly safe to assume that the water in the cracks will be liquid (whether or not air is present in the cracks as well as water is another matter entirely).

[1] The pressure, timescale and propagation speeds were all "given" when the problem was originally posed and we therefore assume them to be correct.

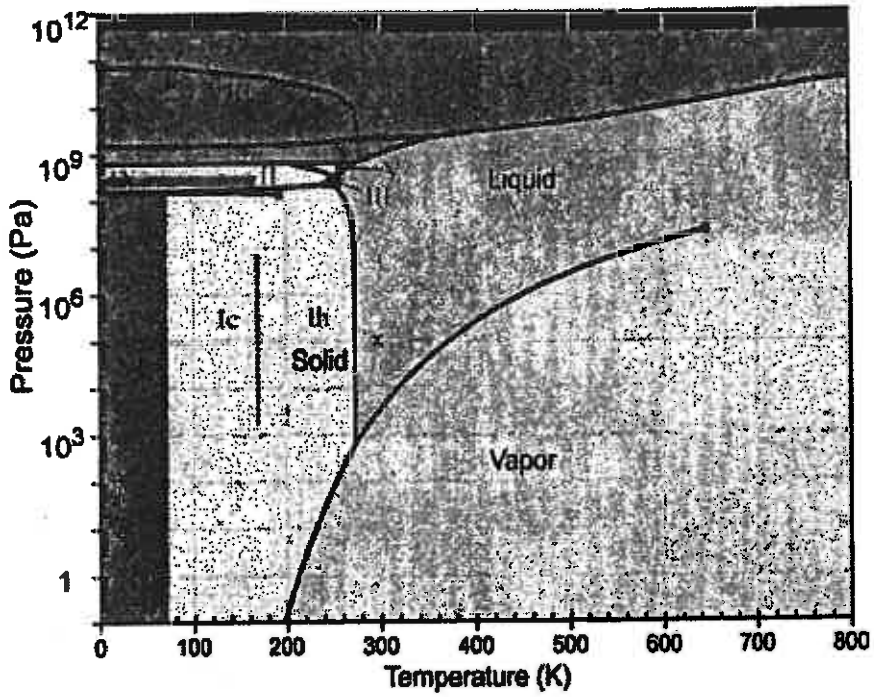

Figure 1: Detailed phase diagram of water showing phase boundaries, triple point and different possible states that water can occupy (reproduced from http://www.lsbu.ac.uk/water/).

Regarding the physical properties of liquid water at the pressures encountered during the cracking process, it is harder than one might anticipate to find accurate data. Some measurements have been carried out, however, and for water at a pressure of 100MPa with $T = 370°\mathrm{C}$ the density and dynamic viscosity are respectively $\rho = 735\mathrm{kg/m^3}$ and $\mu = 8.7 \times 10^{-5}\mathrm{Pa\ s}$, giving a kinematic viscosity of $\nu = 1.18 \times 10^{-7}\mathrm{m^2/s}$ (see, for example, Bridgman (1970)). For the remainder of this report we shall assume that these density and viscosity values are the appropriate ones to use. We also note that the value of the density is reasonably close to the density of water at room temperature: it may further be established that water at these conditions may be regarded as incompressible, and we also assume this from now on.

3 Crack modelling

A number of mathematical models have previously been posed for the process of hydrofracture in rock. Before we can develop a model for the present process, it is necessary to understand the different modelling stances that may be taken to describe fluid-driven cracking. Essentially all modelling

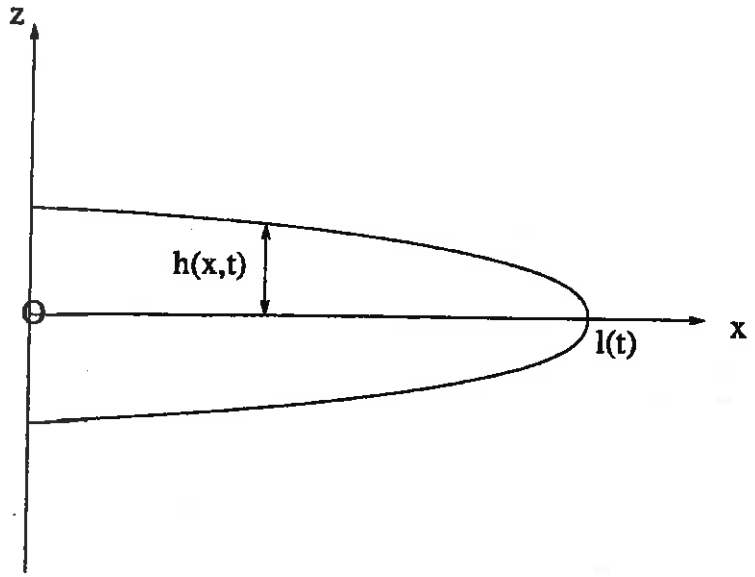

Figure 2: Schematic of nomenclature and coordinate system used for two-dimensional crack modelling (the coordinate direction y points into the page).

approaches may be classified as belonging to one of three paradigms:

- 2D or KGD model. This model was probably first developed in Khristianovic and Zheltov (1955). It assumes that the crack is truly two-dimensional, so that it is identical in every plane y =constant. The crack is also assumed to evolve in a state of plane strain and the fluid flow is independent of y. (For schematic of coordinate system used see Figure 2.)

- PKN model. This model was probably first developed in Nordgren (1972) and Perkins and Kern (1961). Essentially the PKN approach assumes that the cross-sectional shape of the fracture is elliptical. The ellipse is assumed to have constant height H but a width w that decreases as the crack tip is approached. The fluid flow is assumed to be one dimensional in a direction normal to the elliptic cross sections of the fracture. The fact that H is assumed to be constant has led to the PKN model being regarded as particularly appropriate for fractures that are constrained to propagate in the space between two very stiff layers of confining rock, particularly if the strata are very thick or are deep underground. Clearly this model is less relevant in our case than the two-dimensional theory.

- "Penny-shaped" crack. Here it is assumed that the crack is axisym-

metric and takes the form of an expanding disc propagating within a given plane. Fluid is supplied at a known flux and the crack tip is always circular in nature. The normal stress at the crack tip $r = \ell(t)$ is singular like $(\ell - r)^{-1/2}$ and a stress intensity factor may be defined in the normal manner. Though the "penny-shaped" model has enjoyed considerable popularity, it is more appropriate for cases where the injection of fluid is relatively slow, such as at a well-bore or where the shape of the crack is particularly appropriate for this sort of model.

4 Two-dimensional crack modelling

To build a model for the rock-cracking process it will be necessary to consider both the fluid mechanics of the water inside a crack and the way that this interacts with the elasticity of the surrounding rock. We begin by considering the motion of the water inside a crack. (See Figure 2 for nomenclature and details of the coordinates used.)

4.1 Fluid mechanics

We begin by assuming that, as discussed above, the two-dimensional crack model is the most pertinent to the rock fracturing problem that is being considered. We deal first with the fluid mechanics of the problem. The fluid inside the crack obeys the Navier-Stokes equations

$$\boldsymbol{q}_t + (\boldsymbol{q}.\nabla)\boldsymbol{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{q}, \quad \nabla.\boldsymbol{q} = 0\,, \tag{4.1}$$

where t denotes time, $\boldsymbol{x} = (x, z)^T$, $\boldsymbol{q} = (u(x, z, t), w(x, z, t))^T$ is the fluid velocity and the fluid density and kinematic viscosity are denoted respectively by ρ and ν. We also assume that the crack has length $\ell(t)$ and half-width $h(x, t)$. To simplify (4.1) we now non-dimensionalise by setting $x = L\bar{x}$, $z = h_0\bar{z}$, $h = h_0\bar{h}$, $u = U_0\bar{u}$, $w = (U_0h_0/L)\bar{w}$, $t = (L/U_0)\bar{t}$ and $p = (\mu U_0/L\epsilon^2)\bar{p}$. Here L is a typical crack length, h_0 a typical crack half-width, $\epsilon = h_0/L$ is assumed to be small and U_0 is a typical fluid speed in the crack. Note that the scaling for the time t is a matter for debate. For the present we use the standard scaling; this matter will be revisited in Section 7. With these scalings, the non-dimensional equations become (dropping the overbars for simplicity)

$$\epsilon^2 Re(u_t + uu_x + wu_z) = -p_x + \epsilon^2 u_{xx} + u_{zz}\,,$$

$$\epsilon^2 Re(w_t + uw_x + ww_z) = -\frac{p_z}{\epsilon^2} + \epsilon^2 w_{xx} + w_{zz} \,,$$
$$u_x + w_z = 0 \,,$$

where $Re = LU_0/\nu$. It is clear that the two key parameters are ϵ and $\epsilon^2 Re$, so we now examine the size of these two non-dimensional quantities. According to the problem specification, the two extremes of the conditions under which the cracking takes place are given by $U_0 \sim 20$m/s, $P \sim 250$MPa and $U_0 \sim 200$m/s, $P \sim 1000$MPa. Assuming that a typical crack length and width are given by $L = 2$m and $h_0 = 5 \times 10^{-6}$m we then find that the small parameter is given by $\epsilon = 2.5 \times 10^{-6}$. The reduced Reynolds numbers for the two extremes of operative conditions are then $\epsilon^2 Re \sim 0.002$ and $\epsilon^2 Re \sim 0.02$ respectively. It therefore seems completely valid to assume that $\epsilon \ll 1$ and $\epsilon^2 Re \ll 1$. We also note that with these choices the terms p_x and μu_{zz} balance with each other (at both extremes of the range). Finally, it is worth drawing attention to the fact that if the crack size were larger, an alternative approach would have to be employed. For example, lubrication theory would not be valid for cracks of width 0.1mm; in this sense, the theory is sensitive to relatively small changes in order of magnitude of crack size.

The leading order equations are thus those of standard two-dimensional lubrication theory, which, in re-dimensionalised form give

$$p_x = \mu u_{zz}, \quad p_z = 0, \quad u_x + w_z = 0. \tag{4.2}$$

The boundary conditions are that, by symmetry, $u_z = w = 0$ at $z = 0$, and the no-slip condition gives $u = 0$, $w = h_t$ at $z = h(x,t)$. The equations (4.2) may now be solved in the usual way to give

$$u = \frac{p_x}{2\mu}(z^2 - h^2).$$

The next step is to integrate across the crack half-width and use the continuity equation to give

$$\int_0^{h(x,t)} u_x + w_z \, dz = 0.$$

Using u and the boundary conditions now gives the nonlinear evolution equation

$$h_t - \left(\frac{h^3 p_x}{3\mu}\right)_x = 0 \tag{4.3}$$

to determine the crack half-width $h(x,t)$. Of course, (4.3) is not a closed problem as the pressure p has not been determined. To do this we must couple the elasticity of the rock to the motion of the crack.

4.2 Rock elasticity

To provide a closure equation for (4.3) we must consider the elasticity of the rock. Potentially, this introduces many complications, since it is well known that near to a crack tip both plastic and elastic regions are present. We will content ourselves with a linear elastic model that assumes only small displacement gradients in the rock: for the size of crack that we are interested in this seems to be eminently reasonable. Using linear elastic theory, we note (see Muskhelishvili (1953) for full details) that if an elastic half-space $z < 0$ is subjected to respective normal and shear surface stresses $\sigma_{zz} = -p$ and $\sigma_{xz} = q$ then these are related to the surface displacements $U(x)$ and $W(x)$ by

$$-\left(\frac{\mu_s}{1-\nu_p}\right) W_x - \gamma q(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{p(s)}{s-x} ds\,, \tag{4.4}$$

$$\left(\frac{\mu_s}{1-\nu_p}\right) U_x + \gamma p(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{q(s)}{s-x} ds\,, \tag{4.5}$$

where $\gamma = (1-2\nu_p)/(2-\nu_p)$, μ_s is the shear modulus of the elastic material, and ν_p is Poisson's ratio.

The equations (4.4) and (4.5) are completely general and are limited only by the constraints of linear theory: in the case that we wish to consider here, however, to leading order no shear stress is exerted by the crack. Thus $q(x) = 0$ and (4.5) gives the horizontal displacement that the rock must undergo for a given pressure. In contrast, (4.4) gives

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{p(s)}{s-x}\, ds = -\left(\frac{\mu_s}{1-\nu_p}\right) W_x.$$

This full-range Hilbert transform may be inverted (Hilbert transforms of this form are self-inverse save for a sign change) to yield

$$p(x) = \left(\frac{\mu_s}{\pi(1-\nu_p)}\right) \fint_{-\infty}^{\infty} \frac{W_s(s)}{s-x}\, ds.$$

In our case (at least for a symmetrical crack) we have

$$W(x) = \begin{cases} -h(x,t) & (-\ell(t) < x < \ell(t)) \\ 0 & (|\, x \,| > \ell(t)) \end{cases}$$

and if we now assume that we are only interested in cracks that propagate in one direction (so that p and h are undefined for $x < 0$) we find that

$$p(x,t) = -\left(\frac{\mu_s}{\pi(1-\nu_p)}\right) \fint_{0}^{\ell(t)} \frac{h_s(s,t)}{s-x}\, ds. \tag{4.6}$$

4.3 Model closure and boundary conditions

The final model for the crack propagation is therefore given by (4.3) and (4.6), viz

$$h_t - \left(\frac{h^3 p_x}{3\mu}\right)_x = 0\,, \tag{4.7}$$

$$p(x,t) = -\left(\frac{\mu_s}{\pi(1-\nu_p)}\right) \int_0^{\ell(t)} \frac{h_s(s,t)}{s-x}\, ds. \tag{4.8}$$

Though this model was derived independently at MISGSA2004, it is not in fact new and has been used on a number of occasions before. Much discussion at MISGSA2004 revolved around the correct boundary conditions for the problem: it transpires that most of this discussion was incorrect. It is in fact enough to specify three conditions, namely

$$h(x,0) = h_0(x), \quad h(\ell(t),t) = 0, \quad 2\int_0^{\ell(t)} h(x,t)\, dx = Q(t). \tag{4.9}$$

Here $h_0(x)$ denotes the initial crack shape (we implicitly assume the existence of a pre-existing crack of some sort, possibly produced in the first few nanoseconds of the propellant ignition process). The total cross-sectional area of the crack, which is essentially assumed to be known by conservation of mass, is given by $Q(t)$. This final condition may be posed in other guises, but the condition given here is probably the simplest.

Note that one extra boundary condition is *implicit* in the problem, namely that the pressure is finite at $x = 0$. Inspection of (4.8) reveals that, for this to happen, we must have $h_x(0,t) = 0$.

For general h_0 and $Q(t)$ there appears to be no hope of finding closed form solutions to this awkward nonlinear coupled problem. Viewed as a purely numerical problem, it may well be possible to make progress: even this, however, is a far from trivial problem. We shall show in the next section, however, that it is possible to construct similarity solutions that are of interest.

5 Similarity solutions

In order to show that the conditions (4.9) provide enough data to completely solve the problem posed by the equations (4.7) and (4.8) it is worth recalling the work of Spence and Sharp (1985). If we non-dimensionalise (4.7) and (4.8) by setting $t = (3\mu L^3(1-\nu_p)/\mu_s h_0^3)\bar{t}$, $x = L\bar{x}$, $h = h_0\bar{h}$, $\ell = L\bar{\ell}$,

$Q = h_0 L\bar{Q}$ and $p = (h_0\mu_s/L(1-\nu_p))\bar{p}$ then, dropping the overbars for simplicity, we may solve the resulting equations

$$h_t = (h^3 p_x)_x, \quad p = -\frac{1}{\pi}\fint_0^{\ell(t)} \frac{h_s}{s-x}\,ds, \quad 2\int_0^{\ell(t)} h\,dx = Q(t),$$

by assuming a similarity solution of the form

$$p = t^{-1/3}P(\eta), \quad h = kt^{\beta}H(\eta), \quad \ell(t) = kt^{\lambda}, \quad Q = At^{\alpha}, \quad \eta = x/\ell(t).$$

If we then choose $\beta = \frac{1}{2}(\alpha - \frac{1}{3})$ and $\lambda = \frac{1}{2}(\alpha + \frac{1}{3})$, the problem is reduced to

$$\beta H - \lambda\eta H' = (H^3 P')', \quad P(\eta) = -\frac{1}{\pi}\fint_0^1 \frac{H_s}{s-\eta}ds, \quad 2k^2\int_0^1 H(\eta)\,d\eta = A\,,$$

where $'$ denotes a derivative with respect to the similarity variable η. This must now be solved subject to the conditions $H(1) = 0$, $H'(0) = 0$. This problem is still a challenging one, since a singular integral equation coupled to a nonlinear ordinary differential equation must be solved. However, Spence and Sharp (1985) were able to solve this problem numerically by combining the singular integral equation and the differential equation to form a single nonlinear singular integral equation. They then solved this numerically using a spectral method where the unknown function was expressed as a sum of Chebyshev polynomials, which satisfy particularly convenient quadrature rules when combined with a Cauchy kernel. In this way, solutions were calculated and the Stress Intensity Factor could be determined. This similarity approach is probably the closest that one can get to obtaining closed form solutions to the "full" two-dimensional problem.

It should be noted that this is not the only type of similarity solution that is available: Spence and Sharp (1985) showed further that the same problem resulted when one chooses $p = P(\eta)$, $Q = Ae^{\alpha t}$, $\ell(t) = ke^{\lambda t}$ and $h = ke^{\beta t}$, provided only that $\beta = \lambda = \frac{\alpha}{2}$.

6 The "PKN" simplification

As we have seen in Section 4, what may be regarded as the "full" two-dimensional model leads to a very challenging set of equations that defies any attempt at a closed form solution. Even the simplest form of similarity solution leaves us with a distinctly non-trivial singular integral equation to solve.

However, as mentioned in Section 3, an alternative to the classical two-dimensional modelling that has been derived above is available. So-called PKN theory adopts as one of its key premises the fact that the pressure and the crack displacement are related via elasticity in a much more simple way than via (4.8). Various arguments are advanced (see, for example Detournay et al. (2002)) to explain this "new" relationship, but essentially proponents of PKN theory always eventually assume that

$$p = \Lambda h\,, \tag{6.1}$$

where Λ is a constant that may be calculated from the material properties of the rock.

Using (6.1), as an elastic constitutive law for the two-dimensional model, the problem for determining h now becomes the much simpler equation

$$h_t - \left(\frac{\Lambda h^3 h_x}{3\mu}\right)_x = 0. \tag{6.2}$$

The boundary conditions, as before, are given by (4.9), though since the fluid flux is now proportional to h_x it is now no longer appropriate to take $h_x(0,t) = 0$. Instead, the slope at $x = 0$ must be specified to be some non-zero function of time. To analyse (6.2) further, it is convenient to non-dimensionalise by setting $x = L\bar{x}$, $h = h_0\bar{h}$, $t = \tau\bar{t}$, $\ell = L\bar{\ell}$ and $Q = h_0 L\bar{Q}$ where $\tau = (3\mu L^2)/(h_0^3\Lambda)$. Dropping the overbars for convenience, the governing equation becomes

$$h_t = (h^3 h_x)_x. \tag{6.3}$$

The simplest way to express the boundary conditions is to note that

$$\dot{Q} = 2\frac{d}{dt}\left(\int_0^{\ell(t)} h\,dx\right) = \int_0^{\ell(t)} h_t\,dx, \tag{6.4}$$

since $h = 0$ when $x = \ell(t)$. If we now integrate (6.3) with respect to x from 0 to $\ell(t)$, we find, using (6.4) that

$$\dot{Q} = -2h^3(0,t)h_x(0,t).$$

We therefore need to solve (6.3) with $h(\ell(t),t) = 0$, $\dot{Q}(t)$ specified, $h(x,0) = h_0(x)$ and additionally we need to specify either $h(0,t)$ or $h_x(0,t)$, these two quantities being related by

$$h(0,t) = \left(\frac{\dot{Q}(t)}{-2h_x(0,t)}\right)^{1/3}$$

Similarity solutions may now easily be determined. In particular, if we set

$$h(x,t) = t^n f(\eta), \quad \ell(t) = Kt^{-m}, \quad \eta = xt^m ,$$

then we find that the governing partial differential equation is reduced to similarity form so long as $2m + 3n = -1$, in which case it becomes

$$nf - \left(\frac{1+3n}{2}\right)\eta f' = (f^3 f')'.$$

A complete closed-form solution may be obtained when $n = -1/3$, $m = 0$ in which case f is given by

$$\eta = C + \sqrt{\frac{15}{2}} \int^{f(\eta)} \frac{s^3}{\sqrt{D - s^5}}\, ds ,$$

where C and D are arbitrary constants. In this case $\eta = x$, $h(0,t) = t^{-1/3} f(0)$, $h_x(0,t) = t^{-1/3} f'(0)$ and $\dot{Q}(t) \propto t^{-4/3}$.

As one might expect, the physical interpretation of such similarity solutions breaks down when $t = 0$. Note also that, if $n = 0$, it is possible to obtain similarity solutions where $h(0,t)$ is constant (in which case $h_x(0,t) \sim t^{-1/2} f'(0)$). Finally, solutions where $\dot{Q}$ is constant may be determined by choosing $n = \frac{1}{5}$ and $m = -\frac{4}{5}$.

It is also easy to determine the asymptotic behaviour of the solution near to the crack tip $x = \ell(t)$. If we assume that for $x \sim \ell(t)$,

$$h(x,t) \sim (\ell(t) - x)^r ,$$

then substitution into (6.3) soon reveals that $r = 1/3$. As expected, h_x is therefore infinite like $(\ell(t) - x)^{-2/3}$ near to the crack tip.

Such solutions to the "PKN modified" crack model may be further analysed in all of the usual ways: it is possible to determine travelling wave solutions, the behaviour when $\eta \sim 0$ may be determined, and various different behaviours of $h(0,t)$, $h_x(0,t)$ and $\dot{Q}(t)$ may be examined: we do not proceed further with this here.

Finally, it should be pointed out that the "PKN modification" of the full problem suffers, in addition to its somewhat speculative nature, from one severe restriction, namely that by construction the pressure must take the value zero at the crack tip. This completely rules out the possibility of defining a SIF (Stress Intensity Factor) for the spreading crack. Modifications of the "PKN modification" have been proposed to deal with this difficulty, but these are yet more speculative in nature.

7 Impulsive lubrication theory

In Section 4 what might be regarded as a "traditional" lubrication theory approach was followed to model the development of a crack. The ultra high pressures used in the proposed process may nevertheless make it more appropriate to reconsider the derivation of the lubrication theory equations. For the majority of fluid flows the velocity, time and length scales are related by $U = L/T$ and the derivation in Section 4 is correct. However, if impulsive forces are involved in creating fluid flow then different scalings may operate and the time derivative and momentum terms in the Navier-Stokes equations may need to be retained.

To investigate this, we consider the key equation

$$u_t + uu_x + wu_z = -\frac{1}{\rho}p_x + \nu u_{zz}, \tag{7.1}$$

having already discarded the u_{xx} term on the right hand side on the understanding that, whatever the relevant timescales are, we are still concerned with a thin layer flow where z-dimensions are very much less than x-dimensions. If we now non-dimensionalise (7.1) using $x = L\bar{x}$, $z = h_0\bar{z}$, $t = \tau\bar{t}$, $u = U_0\bar{u}$, $w = (h_0U_0/L)\bar{w}$ and $p = (\mu U_0 L/h_0^2)\bar{p}$, then when we rearrange the equation so that its right-hand side is multiplied by unity, we find that the $\bar{u}_{\bar{t}}$ term is multiplied by $h_0^2/\nu\tau$ and the remainder of the left hand side by $h_0^2U_0/\nu L$. The conditions for "impulsive lubrication theory" to be appropriate are therefore

$$\frac{h_0^2}{\nu\tau} \sim 1, \quad \frac{h_0^2U_0}{\nu L} \ll 1, \tag{7.2}$$

under which circumstances the governing equations (4.2) must be replaced by

$$u_t = -\frac{1}{\rho}p_x + \nu u_{zz}, \quad p_z = 0, \quad u_x + w_z = 0. \tag{7.3}$$

When might (7.2) be valid, and might these circumstances be appropriate to the problem considered in this study? If we assume (as we have throughout) that $\nu = 1.18 \times 10^{-7}\mathrm{m}^2/\mathrm{s}$, then we find from (7.2) that the u_t term must be retained whenever $h_0 \geq O(3.4 \times 10^{-4}\sqrt{\tau})\mathrm{m}$. We have noted above the uncertainty in the values of h_0 and τ, but using the given limiting time scales $\tau \sim 0.5\mathrm{ms}$ and $\tau \sim 2.0\mathrm{ms}$ reveals that the u_t term must be retained whenever the cracks have widths between $7.7\mu\mathrm{m}$ and $15\mu\mathrm{m}$ or more. The corresponding values of the size of the uu_x term, given by the second ratio in (7.2), are then about 0.002 and 0.02.

The inescapable conclusion from these estimates appears to be that it is very likely that we should be using (7.3) rather than (4.2).

Of course, using (7.3) complicates matters considerably. The problem that must now be solved merely to determine the flow speed $u(x, z, t)$ consists of (7.3) subject to the conditions

$$u(x, z, 0) = u_0(x, z), \quad u_z(x, 0, t) = 0, \quad u(x, h(x, t), 0) = 0.$$

What started as a simple linear lubrication theory equation has now become a formidable moving boundary problem in three independent variables. The t-dependence of the condition on $z = h(x, t)$ renders all methods such as Laplace transform and separation of variables useless. Unlike standard lubrication theory where time only ever enters the problem as a parameter, a time derivative now appears in the equation of motion and the problem becomes orders of magnitude harder. A small amount of progress may be made by expanding all of the dependent variables as harmonic series in time, but even this produces expressions that would be prohibitively complicated to use in conjunction with (4.6). Only in cases where the pressure has an exponential time dependence can any progress be made. The demanding interplay between the physics of particular forms of loading and appropriate analysis techniques makes impulsive lubrication theory extremely challenging, and both the effects of possible shock wave propagation and the exact nature of fracture initiation need to be taken into account.

Finally, it is worth noting that although much of what was independently derived at MISGSA2004 has in fact previously been known, the work contained in this section is, to the best of our knowledge, completely new. Unfortunately (but predictably) the full problem (4.7), (4.8) is rendered even harder by the inclusion of "impulsive" effects. The equations become so challenging that numerical calculations appear to be the only hope of any progress.

8 Conclusions and recommendations

The following conclusions and recommendations may be listed:

- Before any of the models and calculations contained in this report can be given any real credence more work is required to locate accurate values of the properties of water at the temperatures and pressures encountered in the cracking process. This will probably require a traditional literature search as appropriate and reliable data does not seem to exist electronically.

- Some fairly crude fracture mechanics calculations were carried out at MISGSA2004 to attempt to characterise the initial extent of a crack. These relied on the empirical relationship

$$K = \frac{2\alpha F}{\sqrt{\pi \ell_0}},$$

where K is the stress intensity factor, F is the force per unit length, α is a correlation factor ($\alpha \sim 1.12$) and ℓ_0 the initial length of a crack. We assume that there is no crack if $K^2 < ER_0$ where the quantity $\sqrt{ER_0}$ is known as the "fracture toughness" of the material. When $K^2 = ER_0$, we find that

$$\ell_0 = \frac{4\alpha^2 F^2}{ER_0 \pi}.$$

Assuming that the water is in contact with a length x_c of the inside of the crack and using the fracture toughness for quartzite ($2.2 \times 10^6 \text{Pa}\sqrt{\text{m}}$ at 200MPa) we find that

$$\ell_0 = 13.2 x_c^2$$

where x_c is measured in mm. Thus when say $x_c = 5$mm, the initial crack length is $\ell_0 = 1/3$m. Though this result appears to be intuitively reasonable, more fracture mechanics calculations of this should probably be carried out.

- The boundary conditions for the "full" two-dimensional problem require careful consideration. However, no "extra" condition is required at the crack tip in order to allow the crack to propagate: it is simply enough to set $h\big(\ell(t), t\big) = 0$.

- One way of tackling the "full" two-dimensional problem is via similarity solutions as in Spence and Sharp (1985). This approach leads to a numerical problem that is challenging, but can certainly be solved. Similarity solutions of this type are probably the only way of obtaining solutions to the problem without following a purely numerical approach.

- By adopting the PKN elasticity hypothesis it becomes possible to calculate solutions to the problem with relatively little difficulty. The model is speculative in nature, however, and the constant involved in relating the pressure and the crack displacement will undoubtedly become a "fitting parameter".

- In spite of the progress that can be made when the PKN elasticity hypothesis is adopted, there is a serious objection to this model, namely that the pressure at the crack tip is necessarily zero and so no stress intensity factor can be defined in any meaningful manner.

- Our examination of whether the "impulsive" lubrication theory equations should be used appears to indicate that it is, indeed, necessary to include the u_t term in the governing momentum equation. In discussing this, we have assumed that some of the physical parameters of the problem are fairly well established, while others have been "estimated" using the best means available. If ever more specific information becomes available concerning parameters such as crack length or width, then the order of magnitude analysis performed in Section 7 should be revisited in order to establish whether or not impulsive lubrication theory is appropriate.

- In cases where impulsive lubrication theory is appropriate there seems to be no alternative to proceeding on a solely numerical basis. Though it may be worth expending some more effort to see whether or not any simplifications may be made, it appears unlikely that analytical progress can be made.

- In many such hydrofracture processes fluid leak-off at the crack tip and at other locations on the crack boundary must be accounted for. Various models exist to deal with this complication; the majority propose the inclusion of some sort of sink term in the fluid continuity equation, but all are somewhat speculative in nature. The reader is referred to Detournay et al. (2002) for further details.

- In all of the modelling presented above it has been assumed that the crack contains only water and that no void space is present. It is well known, however, that vapour is often present in such cracks. There are various ways in which the presence of vapour may be taken into account, but these further complicate the modelling and have not been considered in this study.

- As pointed out in Section 2, the role of pre-existing cracks may be key in deciding whether or not the process will be viable as a means of cracking mining rock. The current study did not consider this, but it is clear that any serious model covering the totality of the process will have to assess the importance of pre-existing cracks.

- Finally, we have seen that there are many areas in which further work could be carried out. Further consideration of this problem would make an excellent three-year PHD project which could be jointly supervised between Applied Mathematics and Mechanical Engineering.

Acknowledgements

Thanks are due to Professor Ted Moss for introducing the problem to MISGSA2004 and for his helpful and informative comments. We also acknowledge the contributions that Robert Sarracino (Department of Chemical Engineering, University of Cape Town) made to the problem during the meeting. T.G. Myers and D.P. Mason acknowledge support of this work under the National Research Foundation of South Africa grant numbers 2053289 and 2053745.

References

Bridgman, P.W. (1970). *The Physics of High Pressure.* New York: Dover.

Detournay, E., Adachi, J. and Garagash, D.I. (2002). Asymptotic and intermediate asymptotic behaviour near the tip of a fluid-driven fracture propagating in a permeable elastic medium. In: *Structural Integrity and Fracture*, (Dyskin, Hu and Sahouryeh Eds.), Lisse, Swets and Zeitlinger, pp 9–18.

Khristianovic, S. and Zheltov, Y. (1955). Formation of vertical fractures by means of highly viscous fluids. In: *Proceedings of 4th World Petroleum Congress, Vol 2, (Rome)* pp 579–586.

Muskhelishvili, N.I. (1953). *Some Basic Problems in the Theory of Elasticity.* Groningen, Leyden, Holland: P. Noordhoff, N.V.

Nordgren, R. (1972). Propagation of vertical hydraulic fractures, *J. Petrol. Tech. Trans. AIME*, **253**, 306–314.

Perkins, T. and Kern, L. (1961). Widths of hydraulic fractures. *J. Petrol. Tech. Trans. AIME*, **222**, 937–949.

Spence, D.A. and Sharp, P. (1985). Self-similar solutions for elastohydrodynamic cavity flow. *Proc. Roy. Soc. Lond. A*, **400**, 289–313.